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Packing Rectangles with Polynominoes

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A polyomino of order k is a connected subset of the infinite checkerboard consisting of k of its squares. For a given polyomino, we wish to determine all rectangles which can be packed with copies of this polyomino. Of the polyominos of order up to 4, the only one which cannot pack any rectangle is the N-tetromino shown in Figure 1.

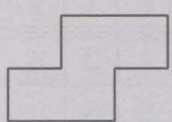

Figure 1

A trivial necessary condition for an order-k polyomino to pack an $m \times n$ rectangle is that k divides mn. The easiest case is when the polyomino is itself a rectangle. The rectangular polyominoes of order up to 4 are shown in Figure 2.

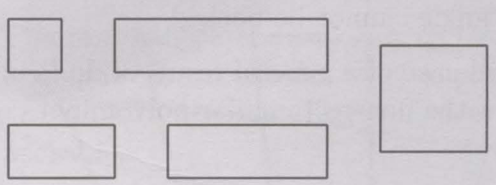

Figure 2

For each of the 1×1 monomino, 1×2 domino and the 1×3 tromino, the trivial necessary condition is also sufficient. This is not the case for the 1×4

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or the 2×2 tetrominoes, since they obviously cannot pack each other. It is easy to see that the 2×2 tetromino packs an $m \times n$ rectangle if and only if 2 divides both m and n.

For the 1×4 tetromino, an $m \times n$ rectangle is packable if 4 divides either m or n. We already know that 4 must divide mn. Suppose 4 does not divide either m or n. Then each of m and n leaves a remainder of 2 when divided by 4. Label the square on the i-th row and the j-th column with $i + j$, reduced modulo 4 to 0, 1, 2 or 3. Figure 3 illustrates the case of a 6×10 rectangle.

2	3	0	1	2	3	0	1	2	3
3	0	1	2	3	0	1	2	3	0
0	1	2	3	0	1	2	3	0	1
1	2	3	0	1	2	3	0	1	2
2	3	0	1	2	3	0	1	2	3
3	0	1	2	3	0	1	2	3	0

Figure 3

In each region of the rectangle, the numbers of squares labelled 0, 1, 2 and 3 respectively are all equal to each other, except in the bottom right corner where there are two 3's and no 1's. No matter how the 1×4 tetromino is placed in this rectangle covering four squares, it must cover one square of each label. Thus this rectangle cannot be packed.

The above is a special case of a general result of de Bruijn [1]. For the rest of this note, we consider the non-rectangular polyominoes of order up to 5 which can pack rectangles.

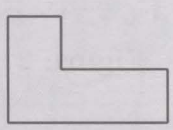

Figure 4

If the L-tetromino in Figure 4 can pack an $m \times n$ rectangle, we know that 4 divides mn. We claim that 8 does also. We may assume that n is even. Colour the n columns alternately yellow and blue. The total number of yellow

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squares is even. Now each copy of the tetromino must cover either 1 or 3 yellow squares. It follows that the number of copies is even, so that 8 indeed divides mn.

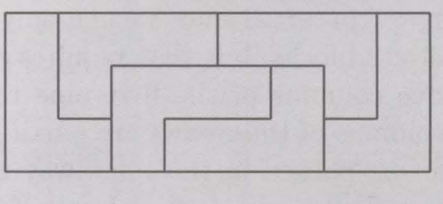

Figure 5

Figure 5 shows that the L-tetromino can pack 2×4 and 3×8 rectangles. Suppose an $m \times n$ rectangle is packable. Since 8 divides mn, we may assume that 4 divides n. If m is even, the rectangle may be partitioned into 2×4 blocks. If m is odd, then 8 must divide n. If $m = 1$, the rectangle is obviously unpackable. For $m \geq 3$, partition the first three rows into 3×8 blocks and the remaining rows into 2×4 blocks. It follows that an $m \times n$ rectangle is packable if and only if 8 divides mn, provided that both m and n are greater than 1. This result is due to Klarner [3].

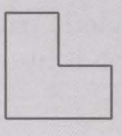

Figure 6

If the V-tromino in Figure 6 can pack an $m \times n$ rectangle, we know that 3 divides mn. We may assume that it divides m. Suppose n is even. We can partition the rectangle into 3×2 blocks. Figure 7 shows that the 2×3 and 5×9 rectangles are packable.

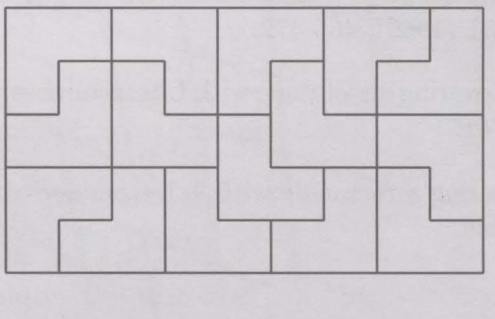

Figure 7

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Suppose n is odd. Clearly, we cannot have $n = 1$. If m is even, the first three columns of the rectangle may be partitoned into 2×3 blocks while the remaining columns may be partitioned into 3×2 blocks. Suppose m is also odd. We cannot have $m = 3$ or $n = 3$ since we can only pack a $3 \times n$ rectangle by partitioning it into 3×2 blocks, but that requires n to be even. If $m \geq 9$ and $n \geq 5$, the first five columns of the first nine rows consist of a 9×5 block. The remaining columns of these rows are partitioned into 3×2 blocks. The remaining rows can be packed in the same way as when m is even. In summary, an $m \times n$ rectangle is packable if and only if 3 divides mn, provided that both m and n are greater than 1, and neither is equal to 3 when the other is odd. This problem is mentioned by Göbel and Klarner [2].

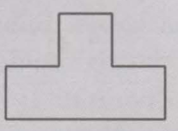

Figure 8

Four copies of the T-tetromino in Figure 8 pack a 4×4 square. If 4 divides both m and n, clearly an $m \times n$ rectangle is packable. An ingenious argument by Walkup [4] shows that this condition is also necessary. We give an elaboration of this in the Appendix.

References

1. N. G. de Bruijn, Filling boxes with bricks, Amer. Math. Monthly **76** (1969) 37–40.
2. F. Göbel & D. A. Klarner, Packing boxes with congruent figures, Indagationes Mathematicae, **31** (1969), 465-472.
3. D. A. Klarner, Covering a rectangle with L-tetrominoes, Amer. Math. Monthly **70** (1963) 760–761.
4. D. Walkup, Covering a rectangle with T-tetrominoes, Amer. Math. Monthly **72** (1965) 986–988.

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Appendix

A packing of a rectangle by copies of the T-tetromino induces a packing of the first quadrant. We divide the quadrant into unit squares using the lines $x = a$ and $y = b$, where a and b are positive integers. We begin with a number of definitions.

A *lattice point* is one whose coordinates are both integral. A *segment* is a line of length 1 joining two lattice points. The *inner segments* of a lattice point are those leading from it towards the left or below. The *outer segments* of a lattice point are those leading from it towards the right or above. We denote by $[x, y]$ the unit square whose bottom left corner is the lattice point (x, y).

An *α-point* is a lattice point (x, y) where $x \equiv y \equiv 0 \pmod 4$ or $x \equiv y \equiv 2 \pmod 4$. A *β-point* is a lattice point (x, y) where $x \equiv 0 \pmod 4$ and $y \equiv 2 \pmod 4$, or $x \equiv 2 \pmod 4$ and $y \equiv 0 \pmod 4$. In subsequent figures, α-points are marked with • while β-points are marked with ∘.

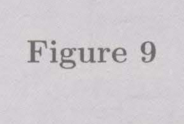

Figure 9

The T-tetromino encompasses ten lattice points and thirteen segments. A *side* of the T-tetromino is a segment which is on its boundary. In Figure 9, only AC, BD and CD are not sides. A *corner* of the T-tetromino is where two of its sides meet at right angles. In Figure 9, only A and B are not corners.

We wish to prove that if an $m \times n$ rectangle is packable with the T-tetromino, then $m \equiv n \equiv 0 \pmod 4$. We claim that this will follow from the following three sequences of statements.

- $R(k)$: The outer segments of an α-point on the line $x + y = 4k$ are sides of some T-tetromino in *every* packing.
- $P(k)$: The inner segments of an α-point on the line $x + y = 4k$ are sides of some T-tetromino in *every* packing.
- $Q(k)$: A β-point on the line $x + y = 4k - 2$ is never a corner of a T-tetromino in *any* packing.

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Suppose $n \not\equiv 0 \pmod 4$. If $n \equiv 2 \pmod 4$, then the β-point $(n, 0)$ is a corner of the rectangle and must therefore be a corner of some T-tetromino. If $n \equiv 1 \pmod 4$, then the outer segments of the α-point $(n-1, 0)$ are sides of some T-tetrominoes. Now the only T-tetromino which can cover the square $[n-1, 0]$ has the β-point $(n-1, 2)$ as a corner. Finally, if $n \equiv 3 \pmod 4$, then the β-point $(n-1, 0)$ dictates that the T-tetromino covering the square $[n-1, 0]$ has its long boundary running from $(n-3, 0)$ to $(n, 0)$. Now the only T-tetromino which can cover the square $[n-1, 1]$ has the β-point $(n-1, 4)$ as a corner.

We now prove the statements by means of simultaneous induction. The basis consists of the statements $R(0)$ and $Q(1)$. The former is trivial since both outer segments in question are on the boundary of the quadrant, and must be sides of some T-tetromino in every packing. No matter how the square [0,0] is covered, neither the β-point (2,0) nor the β-point (0,2) can be a corner of a T-tetromino. The inductive stage is divided into three steps, which are applied cyclically.

First Induction Step. $R(k-1)$ implies $P(k)$ for $k \geq 1$.

If an α-point is on the boundary of the quadrant, it has only one inner segment which is clearly a side of some T-tetromino. We now show that so are the two inner segments of the interior α-point $A(x, y)$ in Figure 10. From the induction hypothesis, we know that the outer segments of the α-points B and C are sides of some T-tetromino.

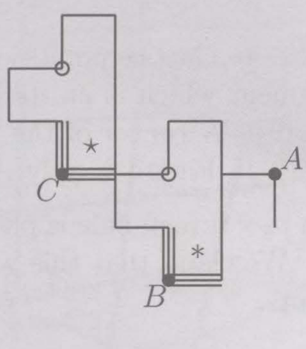

Figure 10

We may assume that the T-tetromino covering the square $[x-2, y-2]$ (marked $*$) has its long boundary vertical. There are three such T-tetrominoes. Two of them have the inner segments of A on their boundaries. The remaining one is shown in Figure 10. Now the square $[x-4, y]$ (marked $\star$) can only be covered as shown, but the new T-tetromino occupies the same relative position as the

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old one. It follows that all translates of it two units to the left and two units up are part of the packing too. However, we run into a contradiction when we get to the y-axis.

Second Induction Step. $Q(k-1)$, $R(k-1)$ and $P(k)$ imply $Q(k)$ for $k \geq 1$.

Suppose $(x,0)$ is a boundary β-point. Then the outer segments of the α-point $(x-2,0)$ and the inner segments of the α-point $(x,2)$ are sides of some T-tetrominoes. The square $[x-1,0]$ can only be covered in two ways, and either makes it impossible for $(x,0)$ to be a corner of any T-tetromino.

We now show that the interior β-point $A(x,y)$ in Figure 15 is not a corner of any T-tetromino. From the induction hypothesis, we know that this is the case with the β-point B. Also, the segments at the α-points E and F, and the inner segments of the α-points C and D are sides of some T-tetrominoes.

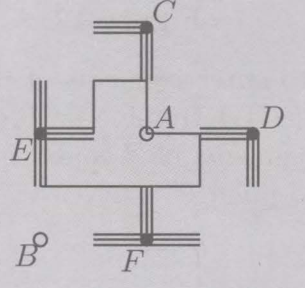

Figure 11

Suppose to the contrary that A is a corner of some T-tetromino. Then two adjacent segments at A are sides of these tetromino. Unless they are the two outer segments, one of the squares $[x-1,y-1]$, $[x-1,y]$ and $[x,y-1]$ cannot be covered. Suppose they are the outer segments as shown in Figure 11. Now the inner segments of A are not sides of any T-tetromino. Hence the square $[x-1,y-1]$ (marked $\star$) is in the same T-tetromino as the squares $[x-1,y]$ and $[x,y-1]$. We may assume that the fourth square of this T-tetromino is $[x-2,y-1]$. Then the only T-tetromino which can cover the square $[x-1,y-2]$ has the β-point B as a corner. This is impossible.

Third Induction Step. $P(k)$ and $Q(k)$ imply $R(k)$ for $k \geq 1$.

The horizontal outer segment of the boundary α-point A in Figure 12 must be a side of some T-tetromino. Suppose the vertical one is not. From the induction hypothesis, we know that the inner segments of the α-points B and C are sides of some T-tetrominoes, while the β-points D and E are never

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corners of any T-tetrominoe. Since D is a β-point, the T-tetrominoes #1 and #2 are forced. They in turn force #3. Since E is a β-point, #2 and #3 force #4, and #3 and #4 in turn force #5. Now #5 is in the same relative position as #3, and we have the same contradiction as in Figure 10.

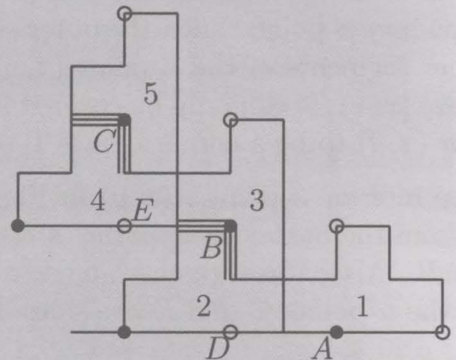

Figure 12

We now show that the two outer segments of the interior α-point $A(x, y)$ in Figure 13 are sides of some T-tetrominoes. From the induction hypotheses, we know that the inner segments of A and the α-point B are sides of some T-tetrominoes, while the β-point C is never a corner of any T-tetrominoe.

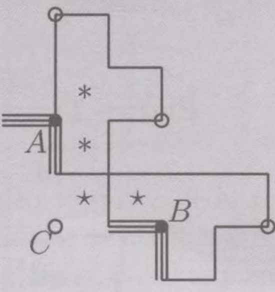

Figure 13

Suppose the horizontal outer segment of A is not a side of a T-tetromino. Then the squares $[x, y]$ and $[x, y-1]$ (marked $*$) belong to the same T-tetromino. There are three such T-tetrominoes, two of which lead to the same contradiction as in Figure 10. The third one is shown in Figure 13. Since C is a β-point, the squares $[x, y-2]$ and $[x+1, y-2]$ (marked $\star$) must belong to different T-tetrominoes, and the one containing $[x+1, y-2]$ is forced by the horizontal inner segment of B. However, this T-tetromino also leads to the same contradiction as in Figure 10. It follows that the horizontal outer segment of A must be a side of some T-teromino, and by symmetry, so is the vertical one.